Acknowledgment

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Boundary Layers for the Conical Navier-Stokes Equations

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Introduction

THE so-called conical Navier-Stokes (N-S) equations are a simplified set of equations that greatly reduce the complexity of numerical computations.[1] The equations are appropriate for studying viscous flows that have truly conical inviscid-flow counterparts, such as supersonic flows past cones and delta wings. They are also useful for generating starting solutions for more complicated parabolized Navier-Stokes computation schemes. Some of the earliest computations were undertaken by McRae[2] and Bluford,[3,4] and current studies also utilize the conical-flow approximation.[5,6] These results show overall good agreement with experimental results and other more complicated numerical schemes. However, it has been noticed that, especially for the thick viscous layers on the leeward side of inclined cones, the conical-approximation results show somewhat thinner boundary layers than indicated by experiment.[2] These discrepancies were attributed to nonconical nose effects associated with the experiments. However, it can be shown for axisymmetric flow that at least the conical N-S equations produce thinner boundary layers than the complete N-S counterparts. Thus it is a shortcoming of the conical N-S equations themselves that their associated boundary layers are too thin. It is the purpose of this Note to demonstrate these results for the simple case of axisymmetric flow.

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Formulation

We consider the simplest case, i.e., viscous axisymmetric supersonic flow past a circular cone (Fig. 1). In general, the flow variables depend on the polar coordinates r and θ. Viscous effects prevail within the boundary layer along the cone surface ($\theta = \theta_c$), in the very thin structure of the shock wave, and in the nose region where r is sufficiently small. In the inviscid region between the boundary layer and the shock, the flow is truly conical, and here the flow variables are functions of θ only.

For the conical N-S equations, it is assumed that the flow variables for the viscous problem also depend on θ only, or, more precisely, that their derivatives with respect to r can be ignored. This is essentially an irrational approximation because the r derivatives of the dependent variables are omitted arbitrarily. The coordinate r still remains in the viscous problem by virtue of the metric scale factors, but it appears only as a parameter and not as a participant in the differentiation on the dependent variables. For the axisymmetric flow case, therefore, only ordinary differential equations involving the polar angle θ occur.

To set up the problem, we denote the r and θ components of velocity by u and v and assume that the pressure p, density ρ, and temperature T are governed by a thermally and calorically perfect gas. Furthermore, we assume that the solution to the associated inviscid problem is known, and we denote the variables for the inviscid solution evaluated at the cone surface θ_c by the subscript e. We shall use these inviscid values to normalize the viscous-flow variables. Thus we introduce the following normalized variables:

$$u/u_e \to u, \qquad v/v_e \to v$$
$$p/p_e \to p, \qquad \rho/\rho_e \to \rho, \qquad T/T_e \to T$$
$$\mu/\mu_e \to \mu \qquad \lambda/\mu_e \to \lambda \tag{1}$$

Here μ and λ are the first and second coefficients of viscosity. To complete the normalization process, we need the nondimensional parameters

$$\epsilon \equiv \frac{\mu_e}{\rho_e u_e r} \tag{2a}$$

$$Pr \equiv \frac{\mu c_p}{k} \tag{2b}$$

$$K_1 \equiv \frac{p_e}{\rho_e u_e^2} \tag{2c}$$

$$K_2 \equiv \frac{c_p T_e}{u_e^2} \tag{2d}$$

where c_p is the constant-pressure specific heat and k the thermal conductivity. We identify ϵ as an inverse characteristic Reynolds number and Pr as the Prandtl number (assumed

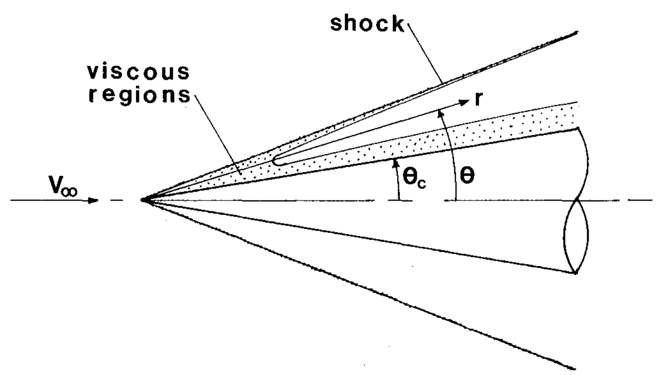

Fig. 1 Schematic for axisymmetric flow past a cone.

constant). In terms of the normalized variables, we can express the steady conical N-S equations for mass, r and θ momentum, and energy as

$$2\rho u \sin\theta \frac{d}{d\theta}(\rho v \sin\theta) = 0 \tag{3}$$

$$\rho v\left(\frac{du}{d\theta} - v\right) = \epsilon\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\mu \sin\theta\left(\frac{du}{d\theta} - v\right)\right\} - D_1\right] \tag{4}$$

$$\rho v\left(\frac{dv}{d\theta} + u\right) = -K_1\frac{dp}{d\theta} + \epsilon\left[\frac{dD_1}{d\theta} - 2(u + v\cot\theta)\frac{d\mu}{d\theta}\right] \tag{5}$$

$$\rho v\frac{dJ}{d\theta} = \frac{\epsilon}{\sin\theta}\left[\frac{d}{d\theta}\left\{\mu \sin\theta\left(u\frac{du}{d\theta} + 2v\frac{dv}{d\theta} + uv + \frac{K_2}{Pr}\frac{dT}{d\theta}\right) + vD_2\sin\theta\right\} + \sin\theta\left\{\mu v\left(\frac{du}{d\theta} - v\right) + uD_2\right\}\right] \tag{6}$$

where

$$J \equiv K_2 T + \tfrac{1}{2}(u^2 + v^2) \tag{7}$$

$$\frac{D_1}{2\mu + \lambda} \equiv \frac{D_2}{\lambda} \equiv 2u\sin\theta + \frac{d}{d\theta}(v\sin\theta) \tag{8}$$

When ϵ vanishes, these equations reduce to the conical inviscid equations of motion.

Boundary Layer

We now consider a set of equations valid near the surface of the cone when ϵ is small, or when the characteristic Reynolds number is large. Based on our experience with classical boundary-layer theory, we introduce a new independent variable measured from the surface of the cone:

$$\theta^* \equiv \frac{\theta - \theta_c}{\sqrt{\epsilon}} \tag{9}$$

Furthermore, we modify the v component of velocity correspondingly, and we get

$$v^* = v/\sqrt{\epsilon} \tag{10}$$

In the boundary layer, we assume that θ^* and v^* are of order unity, as are all of the other variables that were not modified. When the new variables are used and expansions are made for $\epsilon \to 0$, the equations for mass, momentum, and energy reduce to

$$2\rho u + \frac{d}{d\theta^*}(\rho v^*) = \mathcal{O}(\sqrt{\epsilon}) \tag{11}$$

$$\rho v^*\frac{du}{d\theta^*} = \frac{d}{d\theta^*}\left(\mu\frac{du}{d\theta^*}\right) + \mathcal{O}(\sqrt{\epsilon}) \tag{12}$$

$$\frac{dp}{d\theta^*} = \mathcal{O}(\epsilon) \tag{13}$$

$$\rho v^*\frac{dJ_0}{d\theta^*} = \frac{d}{d\theta^*}\left[\mu\left(u\frac{du}{d\theta^*} + \frac{K_2}{Pr}\frac{dT}{d\theta^*}\right)\right] + \mathcal{O}(\sqrt{\epsilon}) \tag{14}$$

where

$$J_0 \equiv K_2 T + \tfrac{1}{2}u^2 \tag{15}$$

We shall consider subsequently the lowest-order boundary-layer equations that are obtained when the terms involving orders of ϵ are ignored.

The lowest-order boundary-layer equations can be placed in a more appropriate form by introducing a new independent variable η by means of the compressibility transformation

$$\eta = \sqrt{2}\int_0^{\theta^*} \rho \, d\theta^* \tag{16}$$

In addition, we introduce the stream function $f(\eta)$ such that

$$u = f'(\eta) \tag{17a}$$

$$\rho v^* = -\sqrt{2}f(\eta) \tag{17b}$$

The lowest-order continuity equation (11) is thus identically satisfied, and the lowest-order r-momentum equation (12) and energy equation (14) reduce to

$$(\rho\mu f'')' + ff'' = 0 \tag{18}$$

$$(\rho\mu J_0')' + PrfJ_0' = (Pr - 1)[\rho\mu f'f'']' \tag{19}$$

Equations (17a), (18), and (19) have precisely the same forms as the corresponding equations obtained from the full N-S equations, and they are governed by the same boundary conditions. The difference lies in the definition of the similarity variable η.

Note to lowest order that the pressure is a constant across the boundary layer, and, therefore, $p = 1$. Thus the thermally perfect equation of state yields $\rho T = 1$. For the linear viscosity-temperature law $\mu = T$, it thus follows that $\rho\mu = 1$, and then Eq. (18) reduces to the classical Blasius equation.

Comparison with N-S Results

For the boundary layer stemming from the complete N-S equations, Eqs. (17a), (18), and (19) also hold, but the similarity variable is defined by[7]

$$\eta_{\text{N-S}} = \sqrt{\frac{3}{2\epsilon}}\int_0^y \rho\frac{dy}{x} \tag{20}$$

where y is measured normal to the cone surface, $x = r$ is measured along the cone surface, and ϵ is defined as before by Eq. (2a). For small angles, we have $\theta - \theta_c = y/x$, and thus η for the conical N-S boundary layer, given by Eq. (16), is related to $\eta_{\text{N-S}}$, for the same y and $x = r$, by

$$\eta = 2/\sqrt{3}\ \eta_{\text{N-S}} \tag{21}$$

Now let us suppose that the edge of the boundary layer $y = \delta$ is defined when $u = 0.99$, for instance, and thus when η_e and $(\eta_{\text{N-S}})_e$ are determined from $f'(\eta_e) = f'[(\eta_{\text{N-S}})_e] = 0.99$. Let the transformed boundary-layer thickness be determined by

$$\Delta = \int_0^\delta \rho \, dy \tag{22}$$

Then, for η and $x = r$ specified, the transformed boundary-layer thicknesses in the two cases are determined by

$$\Delta/x = \sqrt{\epsilon/2}\,\eta_e$$
$$\Delta_{\text{N-S}}/x = \sqrt{2\epsilon/3}\,(\eta_{\text{N-S}})_e \tag{23}$$

Since the edges of the two boundary layers are defined such that $\eta_e = (\eta_{\text{N-S}})_e$, we have for the same location $x = r$ along the cone surface

$$\Delta = \sqrt{3}/2\ \Delta_{\text{N-S}} \cong 0.866\Delta_{\text{N-S}} \tag{24}$$

Thus, for the axisymmetric case, the boundary layer for the conical N-S equations is about 13% thinner than the boundary layer for the complete N-S equations. It can be shown that the same relation as Eq. (24) also holds for the displacement and momentum thicknesses. Since the wall friction and heat transfer are proportional to $f''(0)$, it can be shown that the values for the conical N-S equations are larger than for the complete N-S equations by the factor of $2/\sqrt{3} \cong 1.154$, i.e., by about 15%.

Concluding Remarks

The axisymmetric viscous flows for the conical Navier-Stokes (N-S) equations and the complete N-S equations look very much the same; the boundary layers both being described essentially by the Blasius equation. The similarity scales, however, are different; hence, the boundary layers for the conical N-S equations are thinner. Because the displacement thickness is smaller for the conical N-S equations, the shock location will be displaced outward by a lesser amount than for the complete N-S equations when viscous interaction effects are taken into account, i.e., when ϵ becomes larger (or r smaller). This discrepancy in the shock location, compared with experiment, has also been noticed in Refs. 2 and 3.

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Numerical Evaluation of the Velocities Induced by Trailing Helical Vortices

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Nomenclature

I_u, I_v, I_w = integrals that determine U_i, V_i and W_i, see Eq. (1)
I_u, I_v, I_w = integrands defined in Eqs. (1-3)
I_{ur}, I_{vr}, I_{wr} = remainders used in determining I_u, I_v, and I_w
n = number of blades
p = vortex pitch
$R, R(j,n)$ = lengths defined in Eqs. (4) and (1), respectively
r = radius
U, V, W = velocity in direction of x,y,z in Fig. 1
U_i, V_i, W_i = velocities induced by the trailing helical vortices
x,y,z = co-ordinate directions defined in Fig. 1
β = defined in Eq. (8)
ϵ = absolute error in approximating I_u, I_v, and I_w using remainders
Γ = vortex strength divided by tip radius
θ = angle defined in Fig. 1
θ_m = upper limit for finite integrals

Subscripts

i = point at which induced velocities are required
v = trailing vortex

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Introduction

BOUNDARY integral or "panel" analyses of rotors, propellers, and wind turbines all require the velocities induced by the trailing helical vortices; often these are assumed to have constant pitch. A straight-forward application of the Biot-Savart law then gives U_i, V_i, and W_i in terms of infinite integrals which have no analytic solution. For example, in the co-ordinate system defined in Fig. 1, the axial velocity induced by the n vortices (all of radius r_v) trailing from n blades is given by

$$U_i = \frac{\Gamma}{4\pi} I_u(n) = \frac{\Gamma}{4\pi}\int_0^\infty I_u(n)\,d\theta$$

where

$$I_u(n) = \sum_{j=0}^{n-1} \frac{r_v^2 - r_i r_v \cos(\theta + \theta_v + 2\pi j/n - \theta_i)}{R^3(j,n)} \tag{1}$$

and

$$R^2(j,n) = r_i^2 + r_v^2 - 2r_i r_v \cos(\theta + \theta_v + 2\pi j/n - \theta_i) + (p\theta + x_v - x_i)^2$$

Only one blade and one trailing vortex are shown in Fig. 1 and all lengths are normalized by the tip radius. The V_i and W_i are obtained from Eq. (1) by replacing $I_u(n)$ by $I_v(n)$ and $I_w(n)$, respectively, where

$$I_v(n) = \sum_{j=0}^{n-1}\left[\frac{pz_i - pr_v\cos(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)} - \frac{(p\theta + x_v - x_i) r_v \sin(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)}\right] \tag{2}$$

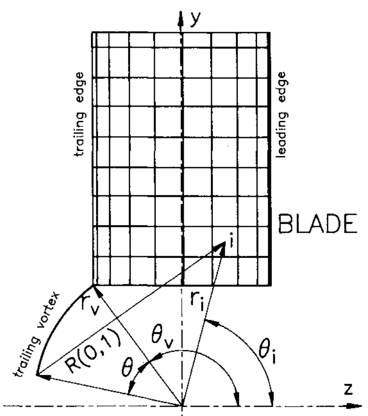

Fig. 1 Coordinate system for wind turbine. View is downstream at blade; the axial or x direction is into page. Typical distribution of panels on lower surface is shown; blade is rotating clockwise.